ON A LINEAR INTEGRAL OF CYCLIC DISPLACEMENT FOR A SOLID BODY ENCLOSING MOBILE MASSES

(O LINEINOM INTEGRALE TSIKLICHESKOGO PEREMESHCHENIIA DLIA TVERDOGO TELA, ZAKLIUCHAIUSHCHEGO VNUTRI SEBIA PODVIZHNYE MASSY)

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1. Knowlege of the cyclic displacement [1] of a mechanical system permits first integrals to be found. This circumstance can afford the possibility of integrating the equations of motion or, at least, of reducing the order of these differential equations. To find the ciclic displacement it is necessary to introduce the S.Lie group of infinitesimal transformations such that at least one (in the best case all) of the group operations would be permuted (commuted) with all the other operations.

Application of the S.Lie group theory methods to find the first integrals has advantage over that of the Lagrange-Routh method , because it permits the detection of mechanical system displacements of more complex structure than the displacements described by the customary Lagrange coordinates. On the other hand, such cyclic displacements simplify matters, permitting first integrals to be found.

Naturally the displacements associated with the Lagrange coordinates are a particular case of the displacements described by the group operations.

2. Let us introduce the S.Lie group of infinitesimal transformations as follows.

Let the position of a solid, which has a fixed point, be defined by nine direction cosines between two rectangular coordinate systems which have a common origin at the fixed point, where one of the systems x_1, x_2, x_3 is fixed in space and the second x^1, x^2, x^3 is coupled invariably to the solid.

Let β_i^k denote these direction cosines between the fixed x_i-axis and the x^k-axis coupled to the solid. Let the quantities $\dot{\beta}_i^k$ (which are mutually dependent) be taken as Poincaré variables [1 and 2]. Let us take Expressions

$$\eta_1 = \frac{1}{2}\left(\frac{\psi'}{a} - \frac{\varphi'}{b}\right), \qquad \eta_2 = \frac{1}{2}\left(\frac{\psi'}{a} + \frac{\varphi'}{b}\right), \qquad \eta_3 = \frac{\theta'}{\sin\theta} \tag{1}$$

as the parameters of the actual displacements.

Here a and b are constants ($a \neq 0$ and $b \neq 0$); the remaining notation is conventional.

Differential dependencies

$$\frac{d\beta_i^k}{dt} = (-1)^k b\beta_i^{3-k}(\eta_1 - \eta_2) + (-1)^i a\beta_{3-i}^k (\eta_1 + \eta_2) + (\beta_i^3\beta_3^k - \delta_i^k)\eta_3$$

$$(i, k = 1, 2, 3) \tag{2}$$

easily verifiable in terms of the Euler angles, may be written for the Poincaré variables.

Here we have put $\beta_i^0 = \beta_0^k = 0$

$$\delta_i^k = \begin{cases} 1 & \text{for} \quad i = k = 3 \\ 0 & \text{in all the other cases} \end{cases}$$

The change in the function of the position of our mechanical system $f(t, \beta_i^k)$ by a real displacement of the system is defined as

$$df = \left(X_0 + \sum_{s=1}^{3} \eta_s X_s f\right) dt$$

where the operations of the S.Lie infinitesimal group of real displacements are

$$X_0 = \frac{\partial}{\partial t}$$

$$X_1 = \sum_{i,k} [(-1)^k b\beta_i^{3-k} + (-1)^i a\beta_{3-i}^k] \frac{\partial}{\partial \beta_i^k}$$

$$X_2 = \sum_{i,k} [(-1)^{3-k} b\beta_i^{3-k} + (-1)^i a\beta_{3-i}^k] \frac{\partial}{\partial \beta_i^k} \tag{3}$$

$$X_3 = \sum_{i,k} (\beta_i^3\beta_3^k - \delta_i^k) \frac{\partial}{\partial \beta_i^k} \qquad (i, k = 1, 2, 3)$$

The values of δ_i^k and β with zero upper or lower index are the same as in Equations (2).

The operations of the subgroup of possible displacements

$$X_1, \quad X_2, \quad X_3 \tag{4}$$

just as the operations of the group of real displacements are permutable, i.e.

$$(X_i, X_k) = 0 \quad (X_i, X_0) = 0 \qquad (i, k = 1, 2, 3) \tag{5}$$

The operations of the subgroup of possible displacements may be written in another form

$$X_1 = a \sum_{i=1}^{3} \left(\beta_1^i \frac{\partial}{\partial \beta_2^i} - \beta_2^i \frac{\partial}{\partial \beta_1^i}\right) + b \sum_{i=1}^{3} \left(\beta_i^1 \frac{\partial}{\partial \beta_i^2} - \beta_i^2 \frac{\partial}{\partial \beta_i^1}\right)$$

$$X_2 = a \sum_{i=1}^{3} \left(\beta_1^i \frac{\partial}{\partial \beta_2^i} - \beta_2^i \frac{\partial}{\partial \beta_1^i}\right) - b \sum_{i=1}^{3} \left(\beta_i^1 \frac{\partial}{\partial \beta_i^2} - \beta_i^2 \frac{\partial}{\partial \beta_i^1}\right) \tag{6}$$

$$X_3 = \sum_{i,k=1}^{3} \beta_i^3\beta_3^k \frac{\partial}{\partial \beta_i^k} - \frac{\partial}{\partial \beta_3^3}$$

3. Let us apply the concept of cyclic displacement to find the first integrals in the problem of the motion of a solid having a fixed point and enclosing mobile masses (a gyroscope or fluid circulating in simply or multiply-connected cavities). This problem was analyzed by Neumann [4], Zhukovskii [5], Sretenskii [6], Kharlamov [7] and others.

The kinetic energy and the force function of the considered mechanical system may be written as

$$T = {}^1\!/_2\, (Ap^2 + Bq^2 + Cr^2) + Pp + Qq + Rr + S, \qquad U = U(\theta, \psi, \varphi) \tag{7}$$

Here, in addition to the conventional notation, we have introduced P, Q, R, S independent of the $\beta_i{}^k$, which characterize the motion of the mobile masses enclosed within the solid. The variables p, q, r may be expressed in terms of the Poincaré variables

$$p = a\beta_3^{\,1}(\eta_1 + \eta_2) + \beta_3^{\,2}\eta_3, \qquad q = a\beta_3^{\,2}(\eta_1 + \eta_2) - \beta_3^{\,1}\eta_3 \tag{8}$$

$$r = a\beta_3^{\,3}(\eta_1 + \eta_2) - b(\eta_1 - \eta_2)$$

Substituting the mentioned values of p, q, r into (7) we obtain the kinetic energy of the system in the Poincaré variables.

$$T = {}^1\!/_2\, \{[a^2\theta + b(b - 2a\beta_3^{\,3})C]\eta_1^2 + [a^2\theta + b(b + 2a\beta_3^{\,3})C]\eta_2^2 +$$

$$+ [A(\beta_3^{\,2})^2 + B(\beta_3^{\,1})^2]\eta_3^2 + 2(a^2\theta - b^2C)\eta_1\eta_2 + 2a(A - B)\beta_3^{\,1}\beta_3^{\,2}\eta_1\eta_3 +$$

$$+ 2a(A - B)\beta_3^{\,1}\beta_3^{\,2}\eta_2\eta_3\} + (aN - bR)\eta_1 + (aN + bR)\eta_2 + (P\beta_3^{\,2} - Q\beta_3^{\,1})\eta_3 + S \tag{9}$$

Here

$$\theta = A(\beta_3^{\,1})^2 + B(\beta_3^{\,2})^2 + C(\alpha_3^{\,3})^2, \qquad N = P\beta_3^{\,1} + Q\beta_3^{\,2} + R\beta_3^{\,3}$$

Of all the Poincaré variables only the quantities β with subscript 3 entered into the expression for the kinetic energy. Hence, only the two members

$$X_1 = b\left(\beta_3^{\,1}\frac{\partial}{\partial\beta_3^{\,2}} - \beta_3^{\,2}\frac{\partial}{\partial\beta_3^{\,1}}\right) + \ldots, \qquad X_2 = -b\left(\beta_3^{\,1}\frac{\partial}{\partial\beta_3^{\,2}} - \beta_3^{\,2}\frac{\partial}{\partial\beta_3^{\,1}}\right) + \ldots$$

will be of value for the calculation of the expressions $X_1(T)$ and $X_2(T)$ in the operations.

The displacements X_1, X_2 will be cyclic according to Chetaev by virtue of (5), which is always satisfied, and of the restrictions

$$X_1(L) = 0, \qquad X_2(L) = 0$$

respectively. These last may be satisfied if we set

$$X_i(T) = 0, \quad X_i(U) = 0 \qquad (i = 1, 2)$$

Condition $X_1(T) = 0$ is satisfied if the constraint

$$A = B, \quad P = Q = 0$$

is imposed on the system.

Let us consider condition $X_1(U) = 0$ which is written in Euler variables as

$$a\frac{\partial U}{\partial\psi} - b\frac{\partial U}{\partial\varphi} = 0 \tag{11}$$

Upon compliance with conditions (10) and (11) we obtain the integral of the cyclic displacement X_1

$$\frac{\partial T}{\partial\eta_1} = [a^2\theta + b(b - 2a\beta_3^{\,3})C]\eta_1 + (a^2\theta - b^2C)\eta_2 + (a\beta_3^{\,3} - b)R = \text{const}$$

In Euler variables this integral will have the form

$$a[A\sin\theta\,(p\sin\varphi + q\cos\varphi) + (Cr + R)\cos\theta] - b(Cr + R) = \text{const} \tag{12}$$

In the particular case with $R = 0$, Goriachev [3] found such an integral for the motion of just a solid around a fixed point under the effect of prescribed forces, by another method.

If it is required that the displacement X_2 be cyclic, the integral may be found which is obtained upon compliance with conditions (10) and $X_2(U) = 0$ which may be written

$$a\frac{\partial U}{\partial\psi} + b\frac{\partial U}{\partial\varphi} = 0$$

This integral has the form

$$a\,[A\sin\theta\,(p\sin\varphi + q\cos\varphi) + (Cr + R)\cos\theta\,] + b\,(Cr + R) = \text{const} \quad (13)$$

The requirement of the simultaneous presence of the cyclic displacements X_1 and X_2 leads to conditions

$$a\frac{\partial U}{\partial\psi} \mp b\frac{\partial U}{\partial\varphi} = 0 \quad (14)$$

(condition (10) is retained here) and to the integrals

$$a\,[A\sin\theta\,(p\sin\varphi + q\cos\varphi) + (Cr + R)\cos\theta] \mp b\,(Cr + R) = \text{const} \quad (15)$$

If the sum and difference of conditions (14) and the integrals (15) are taken, then conditions (14) and the integrals (15) decompose into parts which it is not necessary to write out.

Integration of the equations and study of the motion may furthermore be carried out analogously to the well-studied case of Lagrange motion of a heavy solid.

In addition to the integrals mentioned, the equations of motion have the well known integral of the kinetic energy which does not belong to the category of cyclic displacement integrals.

4. After the cyclic-displacement integral has been found, this integral may be verified simply (12) by using conditions (10) and (11) and the differential equations of motion of the mechanical system

$$A\frac{dp}{dt} + (C - B)\,qr + (Rq - Qr) = \cos\varphi\frac{\partial U}{\partial\theta} + \frac{\sin\varphi}{\sin\theta}\frac{\partial U}{\partial\psi} - \sin\varphi\cot\theta\frac{\partial U}{\partial\varphi}$$

$$B\frac{dq}{dt} + (A - C)\,rp + (Pr - Rp) = -\sin\varphi\frac{\partial U}{\partial\theta} + \frac{\cos\varphi}{\sin\theta}\frac{\partial U}{\partial\psi} - \cos\varphi\cot\theta\frac{\partial U}{\partial\varphi}$$

$$C\frac{dr}{dt} + (B - A)\,pq + (Qp - Pq) = \frac{\partial U}{\partial\varphi}$$

$$\frac{d\theta}{dt} = p\cos\varphi - q\sin\varphi,\quad \frac{d\psi}{dt} = \frac{p\sin\varphi + q\cos\varphi}{\sin\theta},\quad \frac{d\varphi}{dt} = r - \cot\theta\,(p\sin\varphi + q\cos\varphi)$$

The results found may also be expounded for the same parameters of possible displacements by taking Euler angles as Poincaré variables.

The present paper was in press when the Author learned of the work of Keis [8] in which an integral of the form (12) was obtained differently by the method of Goriachev.

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